

Derivatives & integrals

10 essentials · rules, conditions & worked variations

01. Derivative from first principles

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The limit must exist from both sides at an interior point. A corner can be continuous but not differentiable; $|x|$ at 0 is an example.

02. Power & elementary rules

$$(x^n)' = nx^{n-1}, \quad (e^x)' = e^x, \quad (\ln x)' = 1/x$$

Use each rule only where the function is defined and differentiable. For real $\ln x$, $x > 0$. Constants differentiate to zero.

03. Product & quotient

$$(uv)' = u'v + uv', \quad (u/v)' = \frac{u'v - uv'}{v^2}$$

Require $v \neq 0$ for the quotient. Do not multiply derivatives together. Keep the numerator order: derivative of top times bottom, minus the reverse.

04. Chain & implicit differentiation

$$[f(g(x))]' = f'(g(x))g'(x)$$

Differentiate the outside, keep the inside, then multiply by the inside derivative. When y depends on x , differentiating y^2 gives $2yy'$.

05. Tangent & local approximation

$$L(x) = f(a) + f'(a)(x-a)$$

Use near a , where the function is differentiable. The derivative is a rate; the tangent approximation is a predicted function value.

WORKED EXAMPLE 1

Q. For $f(x)=x^2$, find the tangent line at $x=3$.

Answer. $y = 6x-9$. Since $f'(3)=6$ and $f(3)=9$, $y-9=6(x-3)$. The line shares both the value and slope there.

Variation. For $f(x)=x^3$ at $x=2$: slope 12 and tangent $y=12x-16$, because $f(2)=8$.

WORKED EXAMPLE 3

Q. Find the limit of $\sin(3x)/x$ as x approaches 0. Angles are in radians.

Answer. 3. Write it as $3 \times \sin(3x)/(3x)$; the fraction approaches 1.

Variation. For $\sin(5x)/(2x)$, the limit is $5/2$ by the same rule.

CHOOSE THE METHOD Before calculating, check the domain, units, and whether the question asks for a derivative, an accumulated change, or geometric area. A final value alone does not check the method.

WATCH OUT Do not drop the chain-rule factor or $+C$. Critical point does not mean extremum. Use radians in trigonometric derivative rules.

FINAL CHECK

Check the domain first: $\ln(x)$ needs $x>0$; $\ln|x|$ needs $x \neq 0$.

For geometric area, split at sign changes and add absolute areas.

Check an antiderivative by differentiating; check a tangent by substituting its point.

06. Critical points & extrema

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ is undefined}$$

c must be in the domain. For a continuous function on $[a,b]$, compare values at interior critical points and both endpoints for absolute extrema.

07. Second derivative test

$$f'(c) = 0 : \quad f''(c) > 0 \Rightarrow \min, \quad f''(c) < 0 \Rightarrow \max$$

A zero second derivative is inconclusive. An inflection point requires a change in concavity; $f''=0$ alone is not enough.

08. Antiderivatives & substitution

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

For $n=-1$, use $\ln|x|+C$ on an interval avoiding 0. With $u=g(x)$, replace $g'(x)dx$ by du ; change bounds in a definite integral.

09. Fundamental theorem & signed area

$$\int_a^b f(x) dx = F(b) - F(a), \quad F' = f$$

For continuous f on $[a,b]$, an antiderivative evaluates the integral. The integral is signed area; total geometric area uses $|f|$.

10. Integration by parts

$$\int u dv = uv - \int v du$$

Choose u so du simplifies, and dv so v is easy to find. On definite integrals, evaluate uv at the bounds and keep those same bounds.

WORKED EXAMPLE 2

Q. Evaluate $\int$ from 0 to 1 of $2x \cos(x^2) dx$.

Answer. $\sin(1)$. Set $u=x^2$, so $du=2x dx$ and the new bounds are 0 and 1. Then integrate $\cos u$.

Variation. From 0 to $\sqrt{\pi}$ instead: $\sin(\pi)-\sin(0)=0$. Signed areas can cancel even when the curve is not zero.

WORKED EXAMPLE 4

Q. Classify the stationary points of $f(x)=x^3-3x$.

Answer. $x=-1$ is a local maximum; $x=1$ is a local minimum. Here $f'=3x^2-3$ and $f''=6x$.

Variation. For x^3-12x , the points move to -2 (maximum) and 2 (minimum).