

Matrices & linear systems

10 essentials · tests, dimensions & worked variations

01. Dimensions & multiplication

$$A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p} \Rightarrow AB \in \mathbb{R}^{m \times p}$$

Entry $(AB)_{ij} = \sum A_{ik}B_{kj}$: a row meets a column. Inner dimensions must match; AB usually differs from BA.

02. Solving $Ax=b$

consistent $\iff$ no row $[0 \ \cdots \ 0 \mid c]$, $c \neq 0$

Row-reduce the augmented matrix. A pivot in every variable column gives a unique solution if consistent; free variables give infinitely many.

03. Span & linear independence

$Ax=0$ only for $x=0 \iff$ columns independent

Columns span all of $\mathbb{R}^m$ exactly when every row has a pivot. A nonzero solution to $Ax=0$ is a dependence relation among columns.

04. Rank & nullity

$$\text{rank}(A) + \text{nullity}(A) = n$$

n is the number of columns. Pivot columns of the original matrix form a column-space basis; the null space is found from the reduced system.

05. Determinant & inverse

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

For square A , nonzero determinant means full rank and a unique solution for every b . A row swap flips the determinant sign.

WORKED EXAMPLE 1

Q. For $A=\text{diag}(2,3)$, find the eigenvalues and an eigenvector for each.

Answer. $\lambda=2$ with $(1,0)^T$; $\lambda=3$ with $(0,1)^T$. Multiplying scales each coordinate axis by its diagonal entry.

Variation. For $\text{diag}(1,4)$, eigenvalues are 1 and 4 with the same two axis directions. Both matrices are invertible.

WORKED EXAMPLE 3

Q. For $A=[[1,2],[2,4]]$, find the rank, nullity, and whether A is invertible.

Answer. Rank 1, nullity 1, not invertible. The second row is twice the first, so $\det(A)=0$.

Variation. $Ax=0$ gives $x=t(-2,1)$; one free parameter matches nullity 1.

CHOOSE THE METHOD Before solving, write each matrix shape. Check rank before claiming uniqueness, and substitute a proposed solution back into $Ax=b$. A zero determinant rules out a square inverse.

WATCH OUT Check dimensions first. Do not use reduced-matrix columns as a column-space basis. Zero is never an eigenvector; a zero eigenvalue is allowed.

FINAL CHECK

Check dimensions: $(m \times n)(n \times p)$ gives $m \times p$. Reversing the order changes the product.

A projection residual is perpendicular to the target subspace. Check its dot products.

A consistent system has one solution if every variable has a pivot; free variables give parameters.

06. Transpose & inverse order

$$(AB)^T = B^T A^T, \quad (AB)^{-1} = B^{-1} A^{-1}$$

Both formulas reverse order. The inverse rule needs invertible square factors. $(A+B)^{-1}$ is not generally $A^{-1}+B^{-1}$.

07. Eigenvalues & eigenvectors

$$Av = \lambda v, \quad v \neq 0; \quad \det(A - \lambda I) = 0$$

For square A , solve for λ , then find the nonzero vectors in the null space of $A - \lambda I$. A zero eigenvalue means A is singular.

08. Diagonalization

$$A = PDP^{-1}, \quad A^k = PD^k P^{-1}$$

An $n \times n$ matrix is diagonalizable when it has n independent eigenvectors. P holds those vectors; D holds their matching eigenvalues in the same order.

09. Orthogonality & projection

$$\text{proj}_u(v) = \frac{v \cdot u}{u \cdot u} u \quad (u \neq 0)$$

Perpendicular vectors have dot product zero. If the columns of Q are orthonormal, $Q^T Q = I$ and projection onto their span is $QQ^T v$.

10. Least squares

$$A^T A \hat{x} = A^T b, \quad A^T(b - A\hat{x}) = 0$$

Minimize squared error when $Ax=b$ is inconsistent. Full column rank makes $A^T A$ invertible and the minimizer unique; practical solvers often use QR.

WORKED EXAMPLE 2

Q. Project $v=(3,4)$ onto $u=(1,0)$. Is the residual perpendicular to u ?

Answer. Yes. Projection = $(3,0)$, residual = $(0,4)$, and $(0,4) \cdot (1,0) = 0$. Projection keeps the part along the chosen direction.

Variation. For $u=(1,1)$, projection= $(3.5,3.5)$; residual= $(-0.5,0.5)$. Its dot product with u is again zero.

WORKED EXAMPLE 4

Q. Are $u=(1,2)$ and $v=(2,-1)$ perpendicular?

Answer. Yes. Their dot product is $1 \times 2 + 2 \times (-1) = 0$, so the nonzero vectors meet at a right angle.

Variation. Replace v with $(2,1)$: no. The dot product becomes 4, so they are not perpendicular.